The FTC

If f is continuous on $[a,b]$ and $f(t) = F'(t)$, then

$$\int_a^b \frac{dF}{dt}\,dt = \int_a^b f(t)\,dt = \underbrace{\int f(t)\,dt}_{\text{family of antiderivatives of } f(t)} \Big|_{t=a}^{t=b} =$$

$$= \left(F(t)+C\right)\Big|_{t=a}^{t=b} = \left(F(b)+C\right) - \left(F(a)+C\right) =$$

$$= F(b) - F(a)$$

The definite integral of a rate of change gives the total change (or a shorthand notation)

$$\boxed{\int_a^b f(t)\,dt = \underbrace{F(b) - F(a)}_{\left(F(t)+C\right)\big|_{t=a}^{t=b}}}\,, \text{ where } F'(t) = f(t).$$

$$\int_2^3 t\,dt = \frac{t^2}{2}\Big|_{t=2}^{t=3} = \frac{t^2}{2}\Big|_2^3 =$$

$$= \cancel{\tfrac{9}{4}}\ \frac{3^2}{2} - \frac{2^2}{2} = \frac{9-4}{2} = \frac{5}{2}$$

$$\int_5^7 \cos t\,dt = \sin t\Big|_5^7 = \sin(7) - \sin(5)$$

$$\int_5^7 \sin t\,dt = -\cos t\Big|_5^7 = -\cos(7) - (-\cos 5) = \cos(5) - \cos(7)$$

Review of Integration Techniques

① A Collection of Indefinite Integrals we have to remember:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x dx = e^x + C \quad , \quad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{dx}{\cos^2 x} = \tan(x) + C$$

$$\left.\begin{aligned} \int \frac{dx}{1+x^2} &= \arctan(x) + C \\ \int \frac{dx}{\sqrt{1-x^2}} &= \arcsin(x) + C \end{aligned}\right\} \text{(later)}$$

② Guess - and - Check Method

The integrand is almost the same as the integrand from the collection (see part 1). It is the same in sense that x is replaced by $ax+b$. For example

→ $\int \frac{dx}{x-7}$ is almost like $\int \frac{dx}{x} = \ln|x| + C$ → $\begin{cases} \ln(x-7), & x \geq 7 \\ \ln(7-x), & x < 7 \end{cases}$

Thus, we guess that $\int \frac{dx}{x-7} = \ln|x-7| + C$, then we check the answer: $(\ln|x-7|+C)' = \frac{1}{x-7}$.

or we can use the substitution:

$x - 7 = u(x)$
$dx = du$

$$\int \frac{dx}{x-7} = \int \frac{du}{u} = \ln|u| + C = \ln|x-7| + C.$$

→ $\int \cos(ax+b)dx$ is almost $\int \cos x\, dx = \sin x + C$

We guess, that $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + C$.

Check: $\left(\frac{\sin(ax+b)}{a} + C\right)' = \frac{\cos(ax+b) \cdot a}{a} = \cos(ax+b)$

or use the substitution

$ax + b = u(x)$
$a\,dx = du \Rightarrow dx = \frac{du}{a}$.

$$\int \cos(ax+b)dx = \int \cos(u) \frac{du}{a} = \frac{1}{a} \int \cos(u)du = \frac{1}{a}\sin(u) + C =$$
$$= \frac{1}{a} \sin(ax+b) + C.$$

$$\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$$

$\uparrow$

To show that, we used guess-and-check method or the substitution $ax+b=u(x)$.

③ Integration by Parts

This technique is based on the product rule for derivatives. Integration by parts often succeeds with integrals like

$\int x \sin x \, dx$

$\int x e^x dx$

$\int x^2 e^x dx$

$\int \ln x \, dx$ which involve products.

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Goal: To pick u, v in a such way that the RHS integral $\left(\int v \, du\right)$ may be simpler than the LHS integral $\left(\int u \, dv\right)$

Example $\int \sqrt{x} \ln x \, dx$

$\int u \, dv = uv - \int v \, du$

$\sqrt{x} \, dx = dv = v' dx$ $\qquad u = \ln x$

$\sqrt{x} = v'$ $\qquad du = \frac{dx}{x}$

$v = \int \sqrt{x} \, dx = \frac{2x^{\frac{3}{2}}}{3}$

$$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}\cdot\frac{dx}{x} =$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx =$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\cdot x^{\frac{3}{2}}\cdot\frac{2}{3} + C =$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C.$$

$\boxed{\text{Integration of Definite Integrals}}$

When we evaluate a definite integral, the FTC says that we must find an antiderivative of the integrand and then evaluate the antiderivative at the limits of integration.

$F'(t) = f(t)$

$$\int_a^b f(t)\,dt = \int_a^b F'(t)\,dt \overset{\text{FTC}}{=} \int f(t)\,dt\,\Big|_{t=a}^{t=b} =$$

$$= \left(F(t)+C\right)\Big|_{t=a}^{t=b} = \left(F(b)+C\right) - \left(F(a)+C\right) =$$

$$= F(b) - F(a).$$

$$\int_0^4 2x\sqrt{x^2+1}\,dx =$$

$\boxed{\begin{array}{l} \text{substitution} \\ x^2+1 = u(x) \\ 2x\,dx = du \\ \begin{array}{c|c|c} x & 0 & 4 \\ \hline u & 1 & 17 \end{array} \end{array}}$

$$= \int_1^{17} \sqrt{u}\,du = \int_1^{17} u^{\frac{1}{2}}du = \frac{2u^{\frac{3}{2}}}{3}\Big|_{u=1}^{u=17} =$$

$$= \frac{2}{3}(17)^{\frac{3}{2}} - \cancel{3}\frac{2}{3}\cdot(1)^{\frac{3}{2}} = \frac{2}{3}\left[(17)^{\frac{3}{2}}-1\right]$$

$$\int_1^2 \frac{3x^2+1}{x^3+x}\,dx =$$

$\boxed{\begin{array}{l} \text{the inner function is } (x^3+x) \\ x^3+x = u(x) \\ 3x^2+1 = \frac{du}{dx} \Rightarrow (3x^2+1)dx = du \end{array}}$

x	1	2
u	2	10

$$= \int_2^{10} \frac{du}{u} = \ln|u|\Big|_{u=2}^{u=10} = \ln 10 - \ln 2 = \ln 5$$

$$\int_{\frac{1}{2}}^{1} \frac{1}{x^2} e^{\frac{1}{x}}\,dx =$$

$\boxed{\frac{1}{x} = u(x) \qquad -\frac{dx}{x^2} = du \Rightarrow \frac{dx}{x^2} = -du}$

x	$\frac{1}{2}$	1
u	2	1

$$= -\int_2^1 e^u du = \int_1^2 e^u du = e^u\Big|_{u=1}^{u=2} = e^2 - e$$

$$\overset{FTC}{=} \left(\int e^u du\right)\Big|_{u=1}^{u=2} = (e^u + c)\Big|_{u=1}^{u=2}$$

$$\int_0^{\frac{\pi}{6}} \cos x \, e^{\sin x} dx$$

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x \, dx \\ \begin{array}{c|c|c} x & 0 & \frac{\pi}{6} \\ \hline u & 0 & \frac{1}{2} \end{array} \end{array}}$$

$$= \int_0^{\frac{1}{2}} e^u du = e^u \Big|_{u=0}^{u=\frac{1}{2}} = e^{\frac{1}{2}} - 1.$$

$$\int_3^5 x \cos x \, dx =$$

$$\int u \, dv = uv - \int v \, du$$

$$\boxed{\begin{array}{l} \cos x \, dx = dv = v' dx \\ \cos x = v' \\ v = \int \cos x \, dx = \sin x \\ \\ x = u \\ dx = du \end{array}}$$

$$= x \sin x \Big|_{x=3}^{x=5} - \int_3^5 \sin x \, du =$$

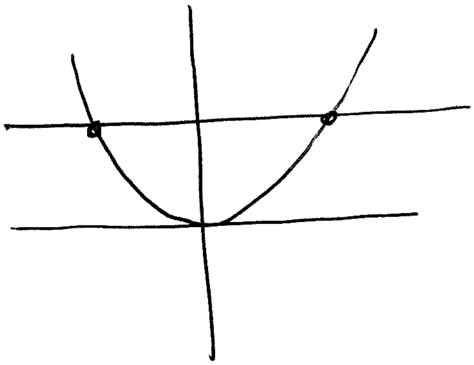
$$= 5 \sin 5 - 3 \sin 3 + \cos x \Big|_3^5 =$$

$$= 5 \sin 5 - 3 \sin 3 + \cos 5 - \cos 3$$

!

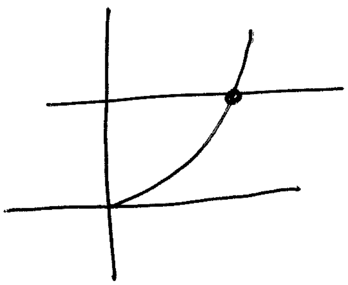
(do not have to change the limits of integration when we use integration by parts technique)

$f(x) = x^2$ has no inverse; because it is not $\boxed{\text{one-to-one}}$ function



$\boxed{\text{Test}}$ The horizontal line test is not satisfied. Since a horizontal line intersects the graph of $f(x)=x^2$ more than one time.

We can make the function invertible by restricting the domain of the function. $(-\infty,\infty)$
Consider only $x \geq 0$ (take a piece of $f(x)$)

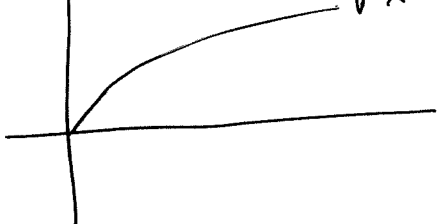


← this function is one-to-one and thus it is invertible

$g(x) = x^2, \ x \geq 0$

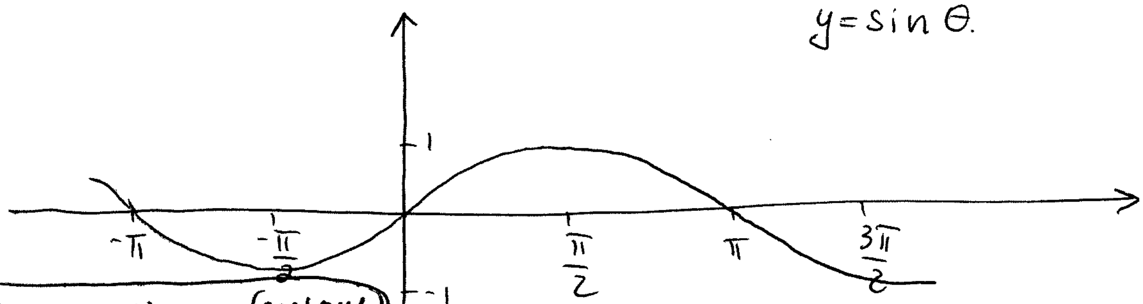
$g^{-1}(x) = \sqrt{x}$; we reflect $g(x) = x^2, \ x \geq 0$ about $y = x$:

$\sqrt{x} = g^{-1}(x)$.



→ The same idea of restriction of the domains of the trigonometric functions allows us to obtain the inverse trigonometric functions: arcsin x, arctan x and so on.

$y = \sin\theta.$



a function: (input) $\longrightarrow$ (output)

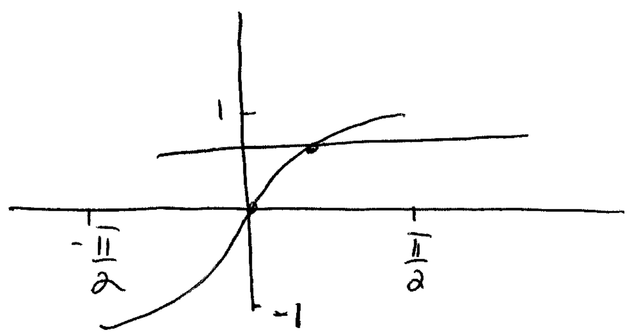
$\sin : \theta \rightarrow y$

θ: an angle, y: a number

$y = \sin\theta$, the domain is $\mathbb{R}$: $-\infty < \theta < \infty$, $\theta \in \mathbb{R}$

the range is $[-1, 1]$, $y \in [-1, 1]$

However, $y = \sin\theta$ is not one-to-one, because the horizontal line test is not satisfied.
Let's restrict the domain. Consider $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

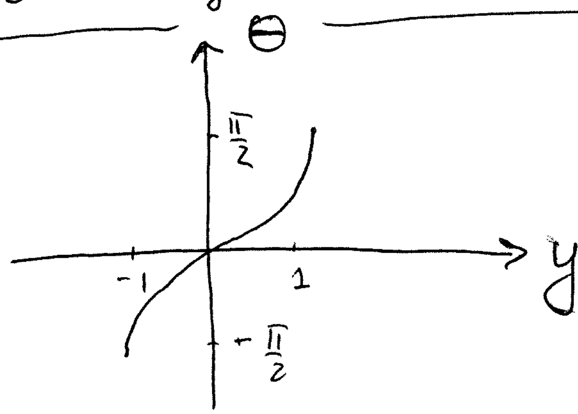


Now, the horizontal test is satisfied and the restricted functio[n] is invertible.

$\sin^{-1}$ or $\arcsin$: $y \rightarrow \theta$

y: a number, θ: an angle

$\theta = \arcsin y$, the domain is $[-1, 1]$
the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



(We reflected the graph of $\sin\theta$ about the diagonal,) Goal:

$(\arcsin y)' - ?$

$\sin(\arcsin y) = y$ ← differentiate both sides with respect to y.

(On the LHS, we use the chain rule)

$(\sin(\arcsin y))' = 1$

$\cos(\arcsin y) \cdot (\arcsin y)' = 1$

$$(\arcsin y)' = \frac{1}{\cos(\underbrace{\arcsin y}_{\theta})} = \frac{1}{\cos(\theta)}$$

Remember: $\cos^2\theta + \sin^2\theta = 1$

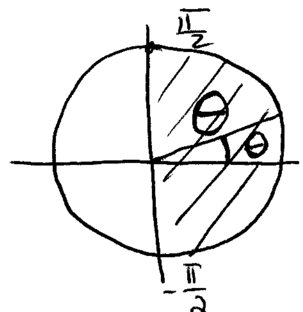
$\cos^2\theta = 1 - \sin^2\theta$

$\cos\theta = \pm\sqrt{1-\sin^2\theta}$

Since we have the restriction on θ:

$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos\theta \geq 0.$



Thus, we exclude $\cos\theta = -\sqrt{1-\sin^2\theta}$ (which is a negative number)

$\cos\theta = \sqrt{1-\sin^2\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Going back to $(\arcsin y)' = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} =$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin y)}} = \frac{1}{\sqrt{1-(\sin(\arcsin y))^2}} = \frac{1}{\sqrt{1-y^2}}$$

$$\boxed{(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}} \Rightarrow \boxed{\int \frac{dy}{\sqrt{1-y^2}} = \arcsin(y) + C}$$